Journal of Mathematical Sciences: Advances and Applications
Volume 1, Number 1, 2008, Pages 109-129

UPDATING PLAYER'S KNOWLEDGE IN THE CONVENTIONAL MODEL IS NOT STRAIGHTFORWARD

TAKEHIRO INOHARA

W9-38, Department of Value and Decision Science
Graduate School of Social and Decision Science and Technology
Tokyo Institute of Technology
2-12-1 O-okayama Meguro
Tokyo 152-8552, Japan
e-mail: inohara@valdes.titech.ac.jp

Abstract

This paper provides an example which indicates that updating player's knowledge is not straightforward in the conventional model of knowledge, and a framework, using players' perceptions of information partitions, which allows us straightforward knowledge updating. An analogue of the concept of common knowledge, called common thought, is proposed, and the equivalence of the concept of common knowledge and that of common thought under the condition of perception of finest common coarsening of the players' information partitions is verified.

1. Introduction

One of the purposes of this current paper is to provide an example which indicates that updating knowledge in Aumann's model [1] of

2000 Mathematics Subject Classification: 91A06, 91A10, 91A12.

Keywords and phrases: knowledge, common knowledge, information partition, hypergame, finest common coarsening.

Received April 30, 2008

knowledge is not straightforward. More concretely, the example shows that straightforward knowledge updating implies that player's knowledge depends on the knowledge of others, which seems to contradict our usual understanding that what a player knows is independent of what others know. To resolve this problem using *players' perceptions of information partitions* is another purpose of this paper. Notation and terminology in this paper are based on [1, 6].

Aumann's framework [1] is used quite commonly to express players' knowledge. In fact, [1] is the seminal work on the notions of knowledge and common knowledge, and [5] is a comprehensive review on the notions of knowledge and common knowledge based on [1]. Vassilakis and Zamir [15] summarize the framework in [1] and clarify the relations between Harsanyi's attribute vectors [7] and the concept of the states of the world in [1, 5] using Mertens and Zamir's universal beliefs space [11]. Topics on backwards induction, common knowledge, and substantial rationality are dealt with in [2, 6, 13] using Aumann's framework [1]. It is shown in [2] that common knowledge of substantial rationality implies the backward induction solution in games of perfect information, whereas in [13] that common knowledge of substantial rationality does not imply the backward induction solution in games of perfect information. Then, the difference of [2] and [13] is discussed and clarified in [6].

An analogue of the concept of common knowledge, called *common thought*, is newly proposed in this paper by using players' perceptions of the information partitions and the concept of strings that is commonly used in hypergame theory [4, 8, 9, 10, 17]. An example that shows difference and similarity between the concept of common knowledge and that of common thought is given in this paper. In particular, it is verified in the main theorem of this paper that common knowledge and common thought are equivalent under the condition of *perception of finest common coarsening of the players' information partitions*.

Although the theory of hypergames treats incompleteness of information in games, it has been developed almost independently of the theory of games of incomplete information. [3] is the seminal work on the theory of hypergames, and [4] gives an overview of the theory of hypergames. [14] developed an operational procedure for conveniently analyzing a hypergame. [16] improved hypergame analysis. [17] gives definitions of solution concepts in hypergames and relationships among them. The hypergame framework is applied also to negotiations [18]. As a generalization of the concept of misperception, the concept of interperception is developed in the seminal works by Inohara [8, 10]. A decision situation is called with *interperception*, if the possibility of the players' misperceptions on the elements of situations are commonly perceived by them. [10] gives a generalization of Nash equilibrium by using the concept of interperception. The difference of the approaches to incompleteness of information in hypergame theory and the theory of games of incomplete information is also discussed in this paper.

In the next section, the conventional model is briefly introduced based on [1, 6]. The example which indicates that updating knowledge is not straightforward is clarified in Section 3, followed by a resolution using players' perceptions of information partitions in Section 4. Section 5 verifies some elementary facts on players' perceptions of information partition, and Section 6 deals with higher-order perceptions and the concept of common thought. Before the conclusions and the discussion on the difference of the approaches to incompleteness of information in hypergame theory and in the theory of games of incomplete information, the main theorem on the equivalence of common knowledge and common thought under the condition of perception of finest common coarsening of the players' information partitions is proved in Section 7.

2. Conventional Model

In this section, a conventional model of *knowledge* is summarized based on [1, 6].

Let $N = \{1, 2, \ldots, n\}$ be the set of all *players*, and let $(\Omega, (P_i)_{i \in N})$ be the pair of the set Ω of all *states* of the world and *information partitions* $(P_i)_{i \in N}$ of the players, where P_i is a partition of Ω for $i \in N$. Usually, P_i is thought to express player i's whole knowledge on the game in which player i is involved, and, in particular, the knowledge should be independent of the knowledge of others, that is, $(P_j)_{j \neq i}$.

For $i \in N$ and $\omega \in \Omega, P_i(\omega)$ is the element of the partition P_i to which state ω belongs. For $i \in N$ and *event* E, that is a subset of Ω, let $K_i(E)$ be $\{\omega \mid P_i(\omega) \subset E\}$. $K_i(E)$ expresses the event that player *i knows* event E. In other words, player i knows event E if the true state ω is in $K_i(E)$. We also say that player i has *knowledge* at state $\omega \in K_i(E)$ that event E occurs.

For $i \in N$ and event E, let $A(E)$ be $\bigcap_{i \in N} K_i(E)$. $A(E)$ expresses that the event that all of the players know event E. In other words, again, all of the players know event E if the true state ω is in $A(E)$. For any integer m such that $m > 1$, $A^m(E)$ denotes $A(A^{m-1}(E))$, where $A^1(E) = A(E)$. Then, for event E, let $CK(E)$ be $\bigcap_{m=1}^{\infty} A^m(E)$. $CK(E)$ expresses the event that event E is *common knowledge* among the players. In other words, again, event E is common knowledge among the players if the true state ω is in $CK(E)$.

As one can easily find from the above definitions, in particular, those of higher-order knowledge such as $K_1(K_2(E))$ for $E \subset \Omega$, that a player's knowledge is defined by involving the knowledge of others. More precisely, the events that a player knows are determined by the information partitions of others as well as by that of his/her own. In the next section, an example indicates that straightforward knowledge updating implies that a player's knowledge depends on the knowledge of others.

3. Example on Updating Knowledge

In this section, a simple example shows that updating knowledge is not straightforward.

Let $\Omega = \{a, b, c\}$. Then, one can have eight possible events, that is, $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$, $\{c, a\}$, and $\{a, b, c\}$. Let $N = \{1, 2\}$, $P_1 = \{\{a, b\}, \{c\}\}$, and $P_2 = \{\{a\}, \{b\}, \{c\}\}$. Then, one can have (first-order) knowledge, that is, $K_1(E)$ and $K_2(E)$ for $E \subset \Omega$, of player 1 and 2, respectively, as in the second and third rows in Table 1. Since $K_1(E)$ and

$K_2(E)$ are events, successive application of K_2 and K_1 to them, respectively, one can have (second-order) knowledge, that is, $K_2(K_1(E))$ and $K_1(K_2(E))$, as in the fifth and fourth rows in Table 1, respectively.

Table 1. First- and second-order knowledge of players 1 and 2 with (P_1, P_2)

$K \setminus E$	$\varnothing$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{b, c\}$	$\{c, a\}$	$\{a, b, c\}$
$K_1(E)$	$\varnothing$	$\varnothing$	$\varnothing$	$\{c\}$	$\{a, b\}$	$\{c\}$	$\{c\}$	{a, b, c}
$K_2(E)$	$\varnothing$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{b, c\}$	$\{c, a\}$	{a, b, c}
$K_1(K_2(E))$	$\varnothing$	$\varnothing$	$\varnothing$	$\{c\}$	$\{a, b\}$	$\{c\}$	$\{c\}$	{a, b, c}
$K_2(K_1(E))$	$\varnothing$	$\varnothing$	$\varnothing$	$\{c\}$	$\{a, b\}$	$\{c\}$	$\{c\}$	{a, b, c}

Now, let us change the information partition of player 2 from P_2 to $P_2' = \{\{a\}, \{b, c\}\}$. That is, player 2 gets to have less knowledge than in above. The same procedure as in the above case with the pair (P_1, P_2) of information partitions derives the (first- and second-order) knowledge in the case with the pair (P_1, P_2') as in Table 2.

Table 2. First-and second-order knowledge of players 1 and 2 with (P_1, P_2')

$K \setminus E$	$\varnothing$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{b, c\}$	$\{c, a\}$	$\{a, b, c\}$
$K_1(E)$	$\varnothing$	$\varnothing$	$\varnothing$	$\{c\}$	$\{a, b\}$	$\{c\}$	$\{c\}$	{a, b, c}
$K_2(E)$	$\varnothing$	$\{a\}$	$\varnothing$	$\varnothing$	$\{a\}$	$\{b, c\}$	$\{a\}$	{a, b, c}
$K_1(K_2(E))$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{c\}$	$\varnothing$	{a, b, c}
$K_2(K_1(E))$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{a\}$	$\varnothing$	$\varnothing$	{a, b, c}

Comparing the fourth rows in Tables 1 and 2, that is, the rows with the index "$K_1(K_2(E))$," one can see that player 1 knows more in Table 1 than in Table 2, in spite of the fact that his/her information partition P_1 is the same in both cases. More concretely, for example, one can see that $K_1(K_2(\{a, b\})) = \{a, b\}$ in Table 1, whereas $K_1(K_2(\{a, b\})) = \emptyset$ in Table 2. The former equation means that player 1 knows the event that player 2 knows event $\{a, b\}$ if the true state is either a or b, whereas the latter equation means that player 1 never knows the event that player 2 knows event $\{a, b\}$ at any true state.

This example seems to contradict with our usual understanding that the information partition of a player describes the player's whole knowledge and that what a player knows is independent of what others, know.

This problem occurs because of the implicit assumption that the players correctly perceive each other's information partition $(P_i)_{i \in N}$, and that the players know which states belong to elements of an information partition. In the next section the author proposes a resolution of this problem by using *players' perceptions of information partitions*.

4. Players' Perceptions of Information Partitions

In this section, the author introduces the concept of *players' perceptions of information partitions* to resolve the problem of *dependence of knowledge* obtained in Section 3. In this section, the terms of 'think' and 'thought' are used for higher-order knowledge instead of 'know' and 'knowledge', respectively, so as not to result any confusions.

Let N, Ω, and $(P_i)_{i \in N}$ be as in Section 2, that is, the set of all players, the set of all states of the world, and information partitions of the players. For partitions P and Q of Ω, P is said to be a *refinement* of Q if $P(\omega) \subset Q(\omega)$ for all $\omega \in \Omega$, denoted by $P \geq Q$. In this case, moreover, Q is said to be a *coarsening* of P.

Consider the cases that each player perceives correctly his/her own information partition but he/she does not always perceive the information partitions of others and he/she has perceptions of those. Let P_j^i denote player i's (first-order) *perception* of P_j, that is, player i perceives that player j has P_j^i as his/her information partitions. It is assumed that $P_i^i = P_i$ for each $i \in N$, because each player usually perceives correctly his/her own information partition. It should be assumed, moreover, that P_j^i is a coarsening of P_i as partitions of Ω, so that the information expressed by P_j^i is included in that by P_i. For $i \in N$, a list $(P_j^i)_{j \in N}$, where $P_i^i = P_i$, is called player i's (first-order) *perceptions of information partitions*, denoted by P^i.

For $i \in N$, $j \in N$, and $\omega \in \Omega$, $P_j^i(\omega)$ is the element of the partition P_j^i to which state ω belongs. If $j = i$, in particular, then $P_j^i(\omega)$ is equal to $P_i(\omega)$, where $P_i(\omega)$ is the element of the partition P_i to which state ω belongs.

For $i \in N$, $j \in N$, and event E, that is a subset of Ω, let $K_j^i(E)$ be $\{\omega \mid P_j^i(\omega) \subset E\}$. $K_j^i(E)$ expresses the event that player i *thinks* that player j knows event E. In other words, player i thinks that player j knows event E if the true state ω is in $K_j^i(E)$. We also say that player i has the *thought* that player j knows event E at state $\omega \in K_j^i(E)$. If $j = i$, in particular, $K_j^i(E)$ is equal to $K_i(E)$, where $K_i(E)$ is defined as $\{\omega \mid P_i(\omega) \subset E\}$, and expresses the event that player i knows event E.

For example, let P_1, P_2^1, and P'^1_2 be $\{\{a\}, \{b\}, \{c\}\}$, $\{\{a, b\}, \{c\}\}$, and $\{\{a, b, c\}\}$, respectively. Note that P_1 is a refinement of P_2^1 and P'^1_2, and that P_2^1 is a refinement of P'^1_2. Then, $K_1(E), K_2^1(E)$, $K_1(K_2^1(E))$, $K'^1_2(E)$, and $K_1(K'^1_2(E))$ are as in the Table 3.

Table 3. Thought of player 1 with P_1, P_2^1, and P'^1_2

$K \setminus E$	$\varnothing$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{b,c\}$	$\{c,a\}$	$\{a,b,c\}$
$K_1(E)$	$\varnothing$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{b,c\}$	$\{c,a\}$	$\{a,b,c\}$
$K_2^1(E)$	$\varnothing$	$\varnothing$	$\varnothing$	$\{c\}$	$\{a,b\}$	$\{c\}$	$\{c\}$	{a, b, c}
$K_1(K_2^1(E))$	$\varnothing$	$\varnothing$	$\varnothing$	$\{c\}$	$\{a,b\}$	$\{c\}$	$\{c\}$	{a, b, c}
$K'^1_2(E)$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	{a, b, c}
$K_2(K'^1_2(E))$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	{a, b, c}

Table 3 shows that the thought derived by P_1 includes that by P_2^1 or that by P'^1_2, in the sense that, at a state, if player 1 thinks by P_2^1 or by P'^1_2 that player 2 knows an event, then player 1 knows the event by P_1 (see Proposition 1 in Section 5). Table 3 also shows that the thought derived by P_2^1 includes that by P'^1_2 (see Lemma 1 and Proposition 1 in Section 5). Moreover, one can see from Table 3 that no additional thought can be derived by P_1 to that by P_2^1 (see Proposition 2 in Section 5). Thus, the problem in the example in Section 3 is overcome, at least in the example given above, using the players' perceptions of information partitions.

The next section is devoted to verifying some elementary facts on the framework proposed in this section, and consequently, to verifying that the above claim in terms of the example that the problem is overcome is generally true.

5. Elementary Facts

In order to verify Propositions 1 and 2, five lemmas should be proved. The proofs of them are almost straightforward.

Lemma 1. *For* $i \in N$, $j \in N$, *and* $\omega \in \Omega$, $P_i(\omega) \subset P_j^i(\omega)$.

Proof. It is evident by the definition of coarsening of partition and the assumption that P_j^i is a coarsening of P_i.

Lemma 1 shows that a coarsening of an information partition derives less thought than the information partition. It supports the relationships of P_2^1 or P'^1_2 toward P_1, and those of P'^1_2 toward P_2^1 in the example in Section 4 (see Table 3 in Section 4).

Lemma 2. *For* $E \subset \Omega$, $K_i(E) \subset E$.

Proof. If $\omega \in K_i(E)$, then $P_i(\omega) \subset E$ by the definition of K_i. Since ω is an element of $P_i(\omega)$, $\omega \in E$ holds.

Lemma 2 shows the event derived by K_i from an event is included in or equal to the original event.

Lemma 3. *For* $\omega \in \Omega$, $K_i(P_i(\omega)) = P_i(\omega)$.

Proofs. $K_i(P_i(\omega)) \subset P_i(\omega)$ holds by Lemma 2. If $\omega' \notin K_i(P_i(\omega))$, then $P_i(\omega')$ is not a subset of $P_i(\omega)$ by the definition of K_i and, in particular, $P_i(\omega') \neq P_i(\omega)$. Since P_i is a partition of Ω, $\omega' \notin P_i(\omega)$.

By Lemma 3, one can see that the event derived by K_i from an element of information partition is equal to the original cell.

Lemma 4. *If* $E \subset F$, *then* $K_i(E) \subset K_i(F)$.

Proof. If $\omega \in K_i(E)$, then $P_i(\omega) \subset E$ by the definition of K_i. If $E \subset F$, then $P_i(\omega) \subset F$ holds, and one can have that $\omega \in K_i(F)$ again by the definition of K_i.

It is verified by Lemma 4 that K_i preserves the relation of "set inclusion" between two sets.

Lemma 5. *For* $E \subset \Omega$, $K_i(K_i(E)) = K_i(E)$.

Proofs. $K_i(K_i(E)) \subset K_i(E)$ holds by Lemma 2. For $\omega \in K_i(E)$, $P_i(\omega) \subset E$ holds by the definition of K_i. By Lemma 4, one can have that $K_i(P_i(\omega)) \subset K_i(E)$. Thus $P_i(\omega) \subset K_i(E)$ by Lemma 3. Therefore, $\omega \in K_i(K_i(E))$ by the definition of K_i.

As in Lemma 3, one can see that the event derived by K_i from $K_i(E)$ is equal to $K_i(E)$. Comparing to Lemma 2, one can see that K_i never shrinks knowledge.

By using these lemmas, the following two properties are verified.

Proposition 1. *For* $E \subset \Omega$, $K_j^i(E) \subset K_i(E)$.

Proof. For $\omega \in K_j^i(E)$, $P_i^j(\omega) \subset E$ holds by the definition of K_j^i. By Lemma 1, one can have that $P_i(\omega) \subset E$, and $\omega \in K_i(E)$ by the definition of K_i.

Proposition 1 means that when player i thinks that player j knows an event at a state, player i also knows the event at the state.

Proposition 2. *For* $E \subset \Omega, K_i(K_j^i(E)) = K_j^i(E)$.

Proof. $K_i(K_j^i(E)) \subset K_j^i(E)$ holds by Lemma 2. $K_i(K_j^i(E)) \supset K_j^i(K_j^i(E))$ also holds by Proposition 1. Since $K_j^i(K_j^i(E)) = K_j^i(E)$ by Lemma 5, one can have that $K_i(K_j^i(E)) \supset K_j^i(E)$.

By Proposition 2, it is implied that all and only the thought of player i on player j's knowledge are described in K_j^i.

6. Higher-Order Perceptions and Common Knowledge

Employing the concept of *strings* of players [4, 8, 9, 10, 17], higher-order perceptions on information partitions and an analogue of the concept of common knowledge, called *common thought* in this paper, can be easily defined and be dealt with.

Let N, Ω, and $(P_i)_{i \in N}$ be as in the Section 2 and 4, that is, the set of all players, the set of all states of the world, and information partitions of the players. Also, let P is the *finest common coarsening* $\wedge_{j \in N} P_j$ of $(P_i)_{i \in N}$, that is, P satisfies (i) $P_i \geq P$ for all $i \in N$, and (ii) if there exists Q such that $P_i \geq Q$ for all $i \in N$, then $P \geq Q$. Moreover, for event E, let $K(E)$ denote $\{\omega \mid P(\omega) \subset E\}$.

For $i \in N$, let, moreover, $\sum_i^*$ be $\{\sigma = i_1 i_2 \cdots i_p \ (p = 1, 2, \ldots) \ | i_{1,} \ i_2, \ldots, i_p \in N, \ i_p = i, \ i_r \neq i_{r+1} (r = 1, 2, \ldots, p-1)\}$ called the set of all strings of players of i. Note that the last player in the string in $\sum_i^*$ is player i in this definition.

Using the strings of players the author gives an inductive definition of players' higher-order perceptions on information partition. For $i \in N$, $j \in N$, and $\sigma = i_1 i_2 \cdots i_p \in \sum_i^*$, P_j^σ denotes player i's (note that $i_p = i$) perception of $P_j^{i_1 i_2 \cdots i_{p-1}}$. That is, P_j^σ means player i's perception of player i_{p-1}'s perception of … player i_1's perception of player j's information partition P_j. Assume that $P_i^i = P_i$ for $i \in N$, and that for $i \in N$, $j \in N$, and $\sigma = i_1 i_2 \cdots i_p \in \sum_i^*$ such that $\sigma \neq i_1, P_j^\sigma = P_j^{i_2 \cdots i_p}$ if $i_1 = j$.

For $i \in N$, $j \in N$, $\sigma = i_1 i_2 \cdots i_p \in \sum_i^*$, and event E, let $K_j^\sigma(E)$ be $\{\omega \mid P_j^\sigma(\omega) \subset E\}$. $K_j^\sigma(E)$ expresses the event that player i thinks that player i_{p-1} thinks that … player i_1 thinks that player j knows event E.

In other words, player i thinks that player i_{p-1} thinks that ... player i_1 thinks that player j knows event E if the true state ω is in $K_j^{\sigma}(E)$. If $j = i_1$, in particular, $K_j^{\sigma}(E)$ is equal to $K_{i_1}^{i_2 i_3 \cdots i_p}(E)$.

For $i \in N$ and $\sigma \in \sum_i^*$, a list $(P_j^{\sigma})_{j \in N}$ is called string σ's *perceptions of information partition*, denoted by P^{σ}, and a list $(P^{\sigma})_{\sigma \in \sum_i^*}$ is called player i's *overall perception of information partitions*, denoted by $\mathbf{P}_i$. One can see that $\mathbf{P}_i$ includes all player i's higher-order perceptions as well as player i's first-order perceptions on information partitions of the player. In order to avoid the problem of *dependence of knowledge* described in Section 3, $\mathbf{P}_i = (P^{\sigma})_{\sigma \in \sum_j^*}$ should satisfy that for $\sigma = i_1 i_2 \cdots i_p \in \sum_j^*$ and $j \in N, P_j^{\sigma}$ is a coarsening of $P_{i_1}^{i_2 i_3 \cdots i_p}$.

Now, let us define an analogure of the concept of common knowledge, called *common thought*. For $i \in N$ and event E, let $A^i(E)$ be $\bigcap_{j \in N} K_j^i(E)$. $A^i(E)$ expresses the event that player i thinks that all of the players know event E. In other words, player i thinks that all of the players know event E if the true state ω is in $A^i(E)$. We also say that player i has the *thought* that all of the players know event E at sate $\omega \in A^i(E)$.

For $i \in N, \sigma = i_1 i_2 \cdots i_p \in \sum_j^*$, and event E, let $A^{\sigma}(E)$ be $\bigcap_{j \in N} K_j^{\sigma}(E)$. $A^{\sigma}(E)$ expresses the event that player i (note that $i_p = i$) *thinks* that player i_{p-1} thinks that ... player i_1 thinks that all of the players know event E. Moreover, for $i \in N$ and event E, let $CT_i(E)$ be $\bigcap_{\sigma \in \sum_i^*} A^{\sigma}(E)$. $CT_i(E)$ expresses the event that player i *thinks* that event E is *common thought* among the players. Then, for event $E, CT(E) = \bigcap_{i \in N} CT_i(E)$ expresses the event that event E is *common* *thought* among the players. In other words, event E is common thought among the players if the true state ω is in $CT(E)$.

Now let us see an example to compare the concept of common knowledge and that of common thought. Let $N = \{1, 2\}$ and $\Omega = \{a, b, c, d\}$. Also, let P_1 and P_2 be $\{\{a, b\}, \{c\}, \{d\}\}$ and $\{\{a\}, \{b\}, \{c, d\}\}$, respectively. Moreover, for $i \in N$ and $\sigma \in \sum_1^*$ such that $\sigma \neq 1$, let P_i^σ be $\{\{a, b\}, \{c, d\}\}$, and for $i \in N$ and $\sigma \in \sum_2^*$ such that $\sigma \neq 2$, let P_i^σ be $\{\{a, b\}, \{c, d\}\}$. Then, one can have player i's overall perceptions of information partitions $\mathbf{P}_i$ for $i \in N$. Then, one can have $CK(E)$, $CT^1(E)$, $CT^2(E)$, and $CT(E)$ for possible event E as in Table 4.

Table 4. Coincidence of common knowledge and common thought

$CK, CT \setminus E$	$\varnothing$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$
$CK(E)$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{a,b\}$	$\varnothing$	$\varnothing$
$CT^1(E)$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{a,b\}$	$\varnothing$	$\varnothing$
$CT^2(E)$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{a,b\}$	$\varnothing$	$\varnothing$
$CT(E)$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{a,b\}$	$\varnothing$	$\varnothing$
$CK, CT \setminus E$	$\{b,c\}$	$\{b,d\}$	$\{c,d\}$	$\{a,b,c\}$	$\{a,b,d\}$	$\{a,c,d\}$	$\{b,c,d\}$	$\{a,b,c,d\}$
$CK(E)$	$\varnothing$	$\varnothing$	$\{c,d\}$	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$	$\{a,b,c,d\}$
$CT^1(E)$	$\varnothing$	$\varnothing$	$\{c,d\}$	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$	$\{a,b,c,d\}$
$CT^2(E)$	$\varnothing$	$\varnothing$	$\{c,d\}$	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$	$\{a,b,c,d\}$
$CT(E)$	$\varnothing$	$\varnothing$	$\{c,d\}$	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$	$\{a,b,c,d\}$

One can see in Table 4 that common knowledge CK and common thought CT coincide with each other. It is because each player has the finest common coarsening of the true information partitions, that is, P_1 and P_2, as his/her perceptions of the other's information partitions (see Theorem 1 in Section 7).

Table 5. Difference of common knowledge and common thought

$CK, CT \setminus E$	$\varnothing$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$
$CK(E)$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{d\}$	$\varnothing$	$\varnothing$	$\{d\}$
$CT^1(E)$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{a,b\}$	$\varnothing$	$\varnothing$
$CT^2(E)$	$\varnothing$	$\{a\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\{a\}$	$\{a\}$	$\{a\}$
$CT(E)$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{a\}$	$\varnothing$	$\varnothing$
$CK, CT \setminus E$	$\{b,c\}$	$\{b,d\}$	$\{c,d\}$	$\{a,b,c\}$	$\{a,b,d\}$	$\{a,c,d\}$	$\{b,c,d\}$	$\{a,b,c,d\}$
$CK(E)$	$\varnothing$	$\{d\}$	$\{d\}$	$\{a,b,c\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{a,b,c,d\}$
$CT^1(E)$	$\varnothing$	$\varnothing$	$\{c,d\}$	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$	$\{a,b,c,d\}$
$CT^2(E)$	$\varnothing$	$\varnothing$	$\varnothing$	$\{a\}$	$\{a\}$	$\{a\}$	$\{b,c,d\}$	$\{a,b,c,d\}$
$CT(E)$	$\varnothing$	$\varnothing$	$\varnothing$	$\{a\}$	$\{a\}$	$\varnothing$	$\{c,d\}$	$\{a,b,c,d\}$

Table 5 is generated by $P_1 = \{\{a, b\}, \{c\}, \{d\}\}$, $P_2 = \{\{a\}, \{b, c\}, \{d\}\}$, $P_i^\sigma = \{\{a, b\}, \{c, d\}\}$ for $i \in N$ and $\sigma \in \sum_1^*$ such that $\sigma \neq 1$, and $P_i^\sigma = \{\{a\}, \{b, c, d\}\}$ for $i \in N$ and $\sigma \in \sum_2^*$ such that $\sigma \neq 2$. In Table 5, common knowledge CK and common thought CT do not coincide anymore. Moreover, there does not exist even the relation of "set-inclusion" between them. This example shows that the concept of common thought is actually different from that common knowledge.

7. Main Theorem

In this section, the main theorem, that claims that the concept of common knowledge and that of common thought coincide under condition of *perception of finest common coarsening of players' information partitions*, is verified. In order to verify the main theorem (Theorem 1), one needs to prove four more lemmas.

Lemma 6. *For* $i \in N$ *and event* E *and* F, *and* $K_i(E) \cap K_i(F) = K_i(F \cap F)$.

Proof. If $\omega \in K_i(E) \cap K_i(F)$, then one can have that $P_i(\omega) \subset E$ and that $P_i(\omega) \subset F$. It is followed by $P_i(\omega) \subset (E \cap F)$. Thus, $\omega \in K_i(E \cap F)$. If $\omega \in K_i(E \cap F)$, then one can have that $P_i(\omega) \subset (E \cap F)$. This means that $P_i(\omega) \subset E$ and that $P_i(\omega) \subset F$. Thus, one can have that $\omega \in K_i(E) \cap K_i(F)$.

Lemma 7. *For event* E, *positive integer* m, *and* $i_1, i_2, \dots i_m \in N$, $A^m(E) \subset K_{i_1}(K_{i_2}(\cdots K_{i_m}(E)))$.

Proof. For $i_1 \in N$, one can have that $A^m(E) = A(A^{m-1}(E)) = \bigcap_{i \in N} K_i(A^{m-1}(E)) \subset K_{i_1}(A^{m-1}(E))$. Similarly, for $i_2 \in N$, one can have that $A^{m-1}(E) = \bigcap_{i \in N} K_i(A^{m-2}(E)) \subset K_{i_2}(A^{m-2}(E))$. By Lemma 4 in Section 5, $K_{i_1}(A^{m-1}(E)) \subset K_{i_1}(K_{i_2}(A^{m-2}(E)))$ is satisfied. By induction, one can have the result.

Before proving Lemma 8, let us define a relation of *reachability* on Ω. For a positive integer m, and ω and ω' in Ω, ω is said to be *reachable* to ω', denoted by $\omega \sim \omega'$, if there exist $\omega_1 (= \omega), \omega_2, \dots, \omega_m (= \omega')$ and $i_1, i_2, \dots i_m \in N$ such that for any $k = 1, 2, \dots, m-1$, $\omega_{i_{k+1}} \in P_{i_k}(\omega_k)$. This definition of reachability and the proof of Lemma 8 are essentially the same as the ones in [12]. Since this relation is an equivalence relation on Ω, it generate a partition of Ω, denoted by $R = \{R(\omega)\}_{\omega \in \Omega}$.

By definition, one can have that for $\omega \in \Omega$, $i \in N$, and $\omega' \in P_i(\omega)$, $\omega \sim \omega'$. Thus, for $i \in N$ and $\omega \in \Omega$, $P_i(\omega) \subset R(\omega)$, that is followed by that R is a coarsening of P_i for all $i \in N$. Therefore, for $\omega \in \Omega$, $\wedge_{j \in N} P_j(\omega) \subset R(\omega)$, since $\wedge_{j \in N} P_j$ is the finest coarsening of $(P_i)_{i \in N}$.

Lemma 8. *Assume that* $P_i^\sigma = \wedge_{j \in N} P_j$ *for all* $i \in N$ *and all* $\sigma \in \sum_i^*$ *such that* $\sigma \neq i$, *where* $\wedge_{j \in N} P_j$ *is the finest coarsening of* $(P_i)_{i \in N}$. *For* $\omega \in \Omega$ *and event E, the following two statements are equivalent*;

1. $\omega \in K_{i_1}(K_{i_2}(\cdots(K_{i_m}(E))))$ *for all positive integer m and all* $i_1, i_2, \ldots i_m \in N$.

2. $\omega \in K_{i_1}^{i_1 i_2 \cdots i_m}(E)$ *For all integer m and all* $i_1, i_2, \ldots i_m \in N$.

Proof. First, assume that $\omega \in K_{i_1}(K_{i_2}(\cdots(K_{i_m}(E))))$ for all positive integer m and all $i_1, i_2, \ldots i_m \in N$. In the case of $m = 1$, in particular, $\omega \in K_{i_1}(E)$ for all $i_1 \in N$. This implies that $\omega \in K_{i_1}^{i_1}(E)$ for all $i_1 \in N$ since $K_{i_1}(E) = K_{i_1}^{i_1}(E)$. in the case of $m > 1$, it has to be shown that $\omega \in K(E)$, because $K_{i_1}^{i_1 i_2 \cdots i_m}(E) = K(E)$ for all positive integer m such that $m > 2$ and all $i_1, i_2, \ldots i_m \in N$ by the assumption of this lemma. $\omega \in K_{i_1}((K_{i_2}(\cdots(K_{i_m}(E))))$ implies that for all $\omega_2 \in P_{i_1}(\omega)$, all $\omega_3 \in P_{i_2}(\omega_2), \ldots,$ and all $\omega_m \in P_{i_{m-1}}(\omega_{m-1})$, $P_{i_m}(\omega_m) \subset E$, which means that $\omega' \in E$ for all ω' such that $\omega \sim \omega'$ with $i_1, i_2, \ldots, i_m$, and some $\omega_1 (= \omega), \omega_2, \ldots, \omega_m (= \omega')$. Therefore, with the consideration on all positive integers m and all $i_1, i_2, \ldots i_m \in N$, one can have that $\omega' \in E$ if $\omega \sim \omega'$. It is followed by $R(\omega) \subset E$. Since $\wedge_{j \in N} P_j(\omega) \subset R(\omega)$ is generally true for $\omega \in \Omega$, $(\wedge_{j \in N} P_j)(\omega) \subset E$, that is, $\omega \in K(E)$, is satisfied.

Next, assume that $\omega \in K_{i_1}^{i_1 i_2 \cdots i_m}(E)$ for all positive integer m and all $i_1, i_2, \ldots i_m \in N$. In the case of $m = 1$, in particular, $\omega \in K_{i_1}^{i_1}(E)$ for all $i_1 \in N$. This implies that $\omega \in K_{i_1}(E)$ for all $i_1 \in N$ since $K_{i_1}^{i_1}(E) = K_{i_1}(E)$. In the case of $m > 1$, $\omega \in K_{i_1}^{i_1 i_2 \cdots i_m}(E)$ means that $\omega \in K(E)$, that is, $(\wedge_{j \in N} P_j)(\omega) \subset E$, by the assumption of this lemma. Since P_{i_1} is a refinement of $\wedge_{j \in N} P_j$, one can have that $P_{i_1}(\omega) \subset (\wedge_{j \in N} P_j)(\omega)$. Since $\omega_2 \in (\wedge_{j \in N} P_j)(\omega)$ for all $\omega_2 \in P_{i_1}(\omega)$ and P_{i_2} is a refinement of $\wedge_{j \in N} P_j$, moreover, one can have that $P_{i_2}(\omega_2) \subset (\wedge_{j \in N} P_j)(\omega)$. Similarly, since $\omega_3 \in (\wedge_{j \in N} P_j)(\omega)$ for all $\omega_3 \in P_{i_2}(\omega)$ and P_{i_3} is a refinement of $\wedge_{j \in N} P_j$, one can have that $P_{i_3}(\omega_3) \subset (\wedge_{j \in N} P_j)(\omega)$. Repeating the same argument, one can have that $P_{i_m}(\omega_m) \subset (\wedge_{j \in N} P_j)(\omega)$ since $\omega_m \in (\wedge_{j \in N} P_j)(\omega)$ for all $\omega_m \in P_{i_{m-1}}(\omega_{m-1})$ and P_{i_m} is a refinement of $\wedge_{j \in N} P_j$. In summary, one can have that for all $\omega_2 \in P_{i_1}(\omega)$, all $\omega_3 \in P_{i_2}(\omega_2), \ldots,$ and all $\omega_m \in P_{i_{m-1}}(\omega_{m-1})$, $P_{i_m}(\omega_m) \subset E$, which means $w \in K_{i_1}(K_{i_2}(\cdots(K_{i_m}(E))))$ for all positive integer m and all $i_1, i_2, \ldots i_m \in N$.

Lemma 9. *For* $\omega \in \Omega$ *and event* $E, \omega \in K_{i_1}(K_{i_2}(\cdots(K_{i_m}(E))))$ *for all positive integer* m *and all* $i_1, i_2, \ldots i_m \in N$ *implies that* $\omega \in A^m(E)$ *for all positive integer* m.

Proof. For $i \in N$ and $i_1, i_2, \ldots i_{m-1} \in N$, one can have that $\omega \in K_{i_1}(K_{i_2}(\cdots(K_{i_{m-1}}(K_i(E)))))$, which means that $\omega \in \bigcap_{i \in N} K_{i_1}$ $(K_{i_2}(\cdots(K_{i_{m-1}}(K_i(E)))))$. Applying Lemma 6 repeatedly, one can have that $\omega \in K_{i_1}(K_{i_2}(\cdots(K_{i_{m-1}}(\bigcap_{i \in N} K_i(E))))) = K_{i_1}(K_{i_2}(\cdots(K_{i_{m-1}}(A(E)))))$. Then, it is satisfied that $\omega \in K_{i_1}(K_{i_2}(\cdots(K_{i_{m-2}}(K_i(A(E))))))$ for $m \in N$ and $i_1, i_2, \ldots i_{m-2} \in N$, which means that $\omega \in \bigcap_{i \in N} K_{i_1}(K_{i_2}$

$(\cdots(K_{i_{m-2}}(K_i(A(E))))))$. Applying Lemma 6 repeatedly again, one can have that $\omega \in K_{i_1}(K_{i_2}(\cdots(K_{i_{m-2}}(\bigcap_{i\in N} K_{i_1}(A(E)))))) = K_{i_1}(K_{i_2}$ $(\cdots(K_{i_{m-2}}(A^2(E)))))$. By induction, one can have that $\omega \in A^m(E)$.

Theorem 1. *Consider* $(N, \Omega, (P_i)_{i\in N}, (\mathbf{P}_i)_{i\in N})$, *where for* $i \in N$, $\mathbf{P}_i = ((P_j^\sigma)_{j\in N})_{\sigma\in\sum_i^*}$ *and* $P_i^i = P_i$. *Then,* $CT(E) = CK(E)$ *for all event* $E \subset \Omega$, *if* $P_i^\sigma = \wedge_{j\in N} P_j$ *for all* $i \in N$ *and all* $\sigma \in \sum_*^*$ *such that* $\sigma \neq i$, *where* $\wedge_{j\in N} P_j$ *is the finest coarsening of* $(P_i)_{i\in N}$.

Proof. Let us assume that $\omega \in CK(E)$. Then, by definition, for all positive integer m, $\omega \in A^m(E)$. By Lemma 7, for all $i_1, i_2, \dots i_m \in N$, $\omega \in K_{i_1}(K_{i_2}(\cdots K_{i_m}(E)))$.

By Lemma 8, $\omega \in K_{i_1}(K_{i_2}(\cdots K_{i_m}(E)))$ for all positive integer m and all $i_1, i_2, \dots i_m \in N$ implies that $\omega \in K_{i_1}^{i_1 i_2 \cdots i_m}(E)$ for all positive integer m and all $i_1, i_2, \dots i_m \in N$, thus it follows that for all $i \in N$ and all $\sigma \in \sum_i^*$, $\omega \in A^\sigma(E)$. Then, one can have that for all $i \in N$, $\omega \in CT_i(E)$, and, by definition, that $\omega \in CT(E)$.

Let us assume that $\omega \in CT(E)$. Then, by definition, one can have that for all $i \in N$, $\omega \in CT_i(E)$. Moreover, also by definition, it is satisfied that for all $i \in N$ and all $\sigma \in \sum_i^*$, $\omega \in A^\sigma(E)$. This implies that for all positive integer m and all $i_1, i_2, \dots, i_m \in N$, $\omega \in K_{i_1}^{i_1 i_2 \cdots, i_m}(E)$. By Lemma 8, $\omega \in K_{i_1}^{i_1 i_2 \cdots, i_m}(E)$ for all positive integer m and all $i_1, i_2, \dots i_m \in N$ implies that $\omega \in K_{i_1}(K_{i_2}(\cdots K_{i_m}(E)))$ for all positive integer m and all $i_1, i_2, \dots, i_m \in N$, thus, by Lemma 9, it follows that for all positive integer m, $\omega \in A^m(E)$. This implies, by definition, that $\omega \in CK(E)$.

8. Conclusions

In this paper, the author provided an example which indicates that updating knowledge is not straightforward in the conventional model of knowledge, and gave a resolution using players' perceptions of information partitions. The validity of the resolution was generally confirmed by one lemma (Lemma 1) and two propositions (Propositions 1 and 2). Whereas the knowledge of a player, in particular, higher-order knowledges in the conventional model is defined by involving the knowledge of others, the thought of a player is defined by involving only the player's perceptions of the information partitions The definition of thought is consistent with our usual understanding that "knowledge" of a player is independent of "knowledge" of others.

The author also treated higher-order perceptions of players and defined the concept of common thought, that was an analogue of the concept of common knowledge. It was verified in Theorem 1 that the concept of common knowledge and that of common thought are equivalent under the condition of perception of finest common coarsening of the players' information partitions.

By an example in Section 6 (see Table 5), it was shown that the concept of common knowledge and that of common thought are mutually different concepts. The difference between the concept of common knowledge and that of common thought reflects the differences of the approaches to incompleteness of information in the theory of games of incomplete information and in the theory of hypergames. One of the critical differences in on the knowledge of the states in an element of an information partition. That is, in the theory of games of incomplete information, a player knows which state belong to an elements of an information partition. Thus, a player is assumed to have probability distribution on the state, and the type space [7] is reasonable to be considered. On the other hand, in the theory of hypergames, a player does not know which states belong to elements of an information partition. So, it is reasonable to assume that a player has his/her perceived game.

Both of the theory of games of incomplete information and the theory for hypergames treat incompleteness of information in games, but they has been developed without interaction. The concepts that were newly proposed and concretely defined in this paper, and possible generalizations of the concept of thought and that of common thought by using the framework of interperception, in particular, the concept of schemes in [8] will connect those two theories, and lead us to a more fruitful theory.

Acknowledgment

The author wishes to express his sincere appreciation to Professor D. Marc Kilgour in Wilfrid Laurier University, who have given valuable comments and suggestions on this paper.

References

[1] R. J. Aumann, Agreeing to disagree, Ann. Statist. 9(6) (1976), 1236-1239.

[2] R. J. Aumann, Backwards induction and common knowledge of rationality, Games and Economic Behavior 8 (1995), 6-19.

[3] P. G. Bennett, Toward a theory of hypergames, OMEGA 5(6) (1977), 749-751.

[4] P. G. Bennett, Hypergames: Developing a model of conflict, Futures 12 (1980), 489-507.

[5] J. Geanakoplos, Common knowledge, R. Aumann and S. Hart (eds.), Handbook of game theory with economic application 2 (Handbooks in economics 11), 1994, pp. 1437-1496, North-Holland, Amsterdam.

[6] J. Y. Halpern, Substantive rationality and backward induction, Games and Economic Behavior 37 (2001), 425-435.

[7] J. Harsanyi, Games of incomplete information played by Bayesian players. Part I, II and III, Management Science 14, 1967-68, pp. 159-182, 320-334, 486-502.

[8] T. Inohara, A formal theory on decision making with interperception, Doctoral dissertation, Tokyo Institute of Technology, Tokyo, Japan, 1997.

[9] T. Inohara, Interperceptional equilibrium as a generalization of Nash equilibrium in games with interperception, IEEE Transactions of systems, Man, and Cybernetics 30(6) (2000), 625-638.

[10] T. Inohara, S. Takahashi and B. Nakano, Integration of games and hypergames generated from a class of games, J. Oper. Res. Soc. 48 (1997), 423-432.

[11] J. F. Mertens and S. Zamir, Formulation of Bayesian analysis for games with incomplete information, International Journal of Game Theory 14 (1985), 1-29.

[12] K. Nishihara, A note on the equivalence of the two definitions of common knowledge, Mathematical Social Sciences 21 (1991), 177-178.

[13] R. C. Stalnaker, Knowledge, belief and counterfactual reasoning in games, Economics and Philosophy 12 (1996), 133-163.

[14] M. A. Takahashi, N. M. Fraser and K. W. Hipel, A procedure for analyzing hypergames, Eur. J. Oper. Res. 18 (1984), 111-122.

[15] S. Vassilakis and S. Zamir, Common belief and common knowledge, Journal of Mathematical Economics 22 (1993), 495-505.

[16] M. Wang, K. W. Hipel and N. M. Fraser, Modeling misperceptions in games, Behavioral Science 33 (1988), 207-223.

[17] M. Wang, K. W. Hipel and N. M. Fraser, Solution concepts in hypergames, Applied Mathematics and Computation 34(3) (1989), 147-171.

[18] M. Wang, R. Liu and Y. Wang, Negotiations with incomplete information, Int. Trans. Oper. Res. 8(6) (2001), 693-705.

■